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Optimal Kernels for a General Sampling Theorem

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1. INTRODUCTION

The well-known sampling theorem (currently ornamented by about a dozen prominent names, see, e.g., [1, 2, 9], and of eminent actual interest in applied signal processing) states that any function (signal) f, the Fourier transform of which has symmetric finite support (i.e., f is bandlimited to) $[-\omega\pi, \omega\pi]$, can be completely reconstructed from its values (samples) $f(k/\omega)$, equally distributed over the real (time) axis, in terms of the cardinal (sampling) series

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\omega}\right) \operatorname{sinc} \pi(\omega t - k) \qquad (\omega \in \mathbb{R}^+,\ t \in \mathbb{R},\ k \in \mathbb{Z}). \tag{1}$$

Its kernel is the famous sinc-function (i.e., sinus cardinalis)

$$\operatorname{sinc} x := x^{-1} \sin x, \qquad \operatorname{sinc} 0 := 1 \qquad (0 \neq x \in \mathbb{R}).$$

During the past hundred years or so many attempts have been made to generalize (1) in a purely mathematical as well as in a practical engineering

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sense. For example, concerning functions which are not a priori band-limited it has been shown that

$$f(t)=\lim_{\omega\to\infty}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{\omega}\right)\operatorname{sinc}\pi(\omega t-k); \tag{2}$$

see [5, 17] as well as [18], the latter giving an extensive list of references with respect to relation (2). Several investigations have also been made by exchanging the sinc-function in (2) to achieve better rates of convergence; see, for instance, [16]. Nevertheless a considerably large number of samples have to be taken into account. This disadvantage can be overcome by replacing the sinc-function by timelimited kernels. In this respect there is a recent general (equivalence) result for generalized sampling series due to Ries and Stens [12].

The purpose of this paper is to restate just an extraction of this theorem, in particular for timelimited kernels, and to construct, mainly as a straightforward application, a simple family of kernels which are optimal (in an almost trivial sense) and which, at the same time, need a minimal number of samples (what is essential for practical purposes). Thus these examples should be not only of inner-mathematical interest but also good enough for real engineering implementation. It might be worthwhile to point out that, apart from familiar ingredients, "central factorial numbers" play an intrinsic part in the proofs.

2. A Sampling Theorem Equipped with High Orders

If the Fourier transform pair (for suitable functions of course) is represented by

$$\varphi^{\wedge}(v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x)\,e^{-ivx}\,dx \multimap \varphi(x)=\int_{-\infty}^{\infty}\varphi^{\wedge}(v)\,e^{ivx}\,dv,$$

the appropriate version of the above mentioned general theorem may be formulated as follows.

PROPOSITION. *Let $\varphi\in C(\mathbb{R})$ have finite support $[-\sigma,\sigma]$, $\sigma\in\mathbb{R}^+$ with $2\sigma\in\mathbb{N}$, and let there exist $r\in\mathbb{N}$, $r\geqslant 2$, such that for $\varphi^{\wedge}$ and its derivatives it holds that*

$$\varphi^{\wedge}(2k\pi)=\delta_{k,0}\qquad (k\in\mathbb{Z}), \tag{3}$$

$$D^j\varphi^{\wedge}(2k\pi)=0,\ 1\leqslant j\leqslant r-1\qquad (j\in\mathbb{N}). \tag{4}$$

Then for $f \in C^{(r)}(\mathbb{R})$ it follows that

$$\left\| \sum_{k=[\omega\circ-\sigma]+1}^{[\omega\circ+\sigma]} f\left(\frac{k}{\omega}\right) \varphi(\omega\circ-k) - f(\circ) \right\|_C \leqslant \frac{M}{\omega^r} \quad (\omega \in \mathbb{R}^+); \tag{5}$$

the maximal number of terms in the sum is exactly 2σ, the right-hand constant being explicitly given (roughly) by

$$M := \frac{2}{r!} \|\varphi\|_C \|f^{(r)}\|_C \sigma^{r+1}. \tag{6}$$

(As usual, the Gaussian brackets $[x]$ denote the largest integer $\leqslant x \in \mathbb{R}$.)

The proof proceeds via a known technique using the Poisson summation formula (see, e.g., [4, pp. 201; 194]): for $\lambda \in \mathbb{Z}$, $0 \leqslant \lambda \leqslant r-1$,

$$\sum_{k=-\infty}^{\infty} (u-k)^\lambda \varphi(u-k) = \sum_{k=-\infty}^{\infty} (-i)^\lambda D^\lambda \varphi\hat{}\,(2k\pi) e^{-i2k\pi u} \qquad (u \in \mathbb{R}).$$

It is read off that conditions (3) and (4), respectively, are equivalent to

$$\sum_{k=-\infty}^{\infty} \varphi(u-k) = 1, \quad \sum_{k=-\infty}^{\infty} (u-k)^j \varphi(u-k) = 0 \qquad (1 \leqslant j \leqslant r-1).$$

Hence the Taylor series expansion

$$f\left(\frac{k}{\omega}\right) = \sum_{j=0}^{r-1} \frac{f^{(j)}(t)}{j!} \left(\frac{k}{\omega} - t\right)^j$$
$$+ \frac{f^{(r)}(\xi)}{r!} (k-\omega t)^r \omega^{-r} \qquad (\xi = \xi(k, \omega, t))$$

immediately leads (note vanishing terms) to

$$\sum_{k=-\infty}^{\infty} f\left(\frac{k}{\omega}\right) \varphi(\omega t - k) - f(t)$$
$$= \omega^{-r} \sum_{k=-\infty}^{\infty} \frac{f^{(r)}(\xi)}{r!} (k-\omega t)^r \varphi(\omega t - k).$$

Indeed, since $\varphi(x) \equiv 0$, $x \notin [-\sigma, \sigma]$, both infinite series reduce to finite sums with generally 2σ terms and $[\omega t - \sigma] + 1 \leqslant k \leqslant [\omega t + \sigma]$; only in the boundary case with $(\omega t - \sigma) \in \mathbb{N}$ one further term cancels out. Finally, assertion (5) is an easy consequence, the constant of (6) being derived from

$$\sum_{\lambda=1}^{2\sigma} |[\omega t - \sigma] + \lambda - \omega t|^r |\varphi(\omega t - [\omega t - \sigma] - \lambda)|$$

noting that $|[\omega t - \sigma] - \omega t + \lambda| \leqslant \sigma$, $\lambda = 1, 2, \ldots, 2\sigma$.

Summarizing: this sampling theorem with arbitrarily increasing approximation order $O(\omega^{-r})$, $\omega \to \infty$, $r \geqslant 2$ needs just the very restricted number of at most 2σ samples (provided the time limit σ is small enough). In fact, it will be shown that there exist (simple) sampling kernels φ which satisfy the foregoing proposition, and that for small σ.

3. Realization

Let i, $n \in \mathbb{N}$, and for $1 \leqslant i \leqslant n$ let $p_i \in \mathbb{N}$ such that $2 \leqslant p_1 < p_2 < \cdots < p_n$, in short $\not{p} := (p_1, p_2, ..., p_n)$. For $\gamma_i \in \mathbb{R}$, $\gamma := (\gamma_1, \gamma_2, ..., \gamma_n)$, define the linear combination ϕ_n of n (central) B-splines by

$$\phi_n(x) \equiv \phi_n(x; \not{p}; \gamma) := \sum_{i=1}^{n} \gamma_i M_{p_i}(x) \qquad (x \in \mathbb{R}) \tag{7}$$

these basic splines being given as the Fourier transforms of the powers of the sinc-function (in other words, of the general Jackson–de La Vallée Poussin kernels [20] of the theory of singular integrals on the line), namely,

$$\begin{aligned} M_n(x) &:= \frac{1}{\pi} \int_0^{\infty} \left(\operatorname{sinc} \frac{t}{2}\right)^n \cos xt \, dt \qquad (n \geqslant 2) \\ &= \begin{cases} \dfrac{1}{(n-1)!} \displaystyle\sum_{k=0}^{[(n/2)-|x|]} (-1)^k \binom{n}{k} \left(\frac{n}{2} - |x| - k\right)^{n-1}, & |x| \leqslant \dfrac{n}{2}, \\ 0, & |x| \geqslant \dfrac{n}{2}; \end{cases} \end{aligned} \tag{8}$$

see, e.g., [7; 8, p. 457, misprint!]. Thus, as is well known (see, e.g., [14, p. 11f]), M_n has symmetric finite support $[-n/2, n/2]$, is a *piecewise polynomial* of degree $n-1$ on that interval, and has a continuous derivative of order $n-2$.

Since by the inverse Fourier transform

$$M_n^{\wedge}(v) = 2 \int_0^{\infty} M_n(x) \cos vx \, dx = \left(\operatorname{sinc} \frac{v}{2}\right)^n \qquad (v \in \mathbb{R}) \tag{9}$$

the transform of (7) is given by

$$\phi_n^{\wedge}(v) \equiv \phi_n^{\wedge}(v; \not{p}; \gamma) = \sum_{i=1}^{n} \gamma_i \left(\operatorname{sinc} \frac{v}{2}\right)^{p_i}. \tag{10}$$

The purpose now is to determine the n-tuple of coefficients γ such that the conditions of the proposition are satisfied for the actual transform pair ϕ_n, $\phi_n^{\wedge}$. In this respect there is

THEOREM 1. *For* (10) *there holds the Taylor series expansion*

$$\phi_n^{\wedge}(v; \not{p}; \gamma^*) = 1 - \frac{1}{2^{3n} \cdot 3^n \cdot n!} \prod_{\mu=1}^{n} p_\mu \cdot v^{2n} + O(v^{2n+2}) \qquad (v \to 0) \tag{11}$$

(*with all but the first of the initial coefficients vanishing*) *if and only if*

$$\gamma_i^* = (-1)^{n+1} {\prod_{\mu=1}^{n}}' \frac{p_\mu}{p_i - p_\mu} \in \mathbb{Q} \qquad (1 \leqslant i \leqslant n) \tag{12}$$

(*the prime indicating that the index* $\mu = i$ *is to be excluded*). *Moreover, with respect to the zeroes of* (10) *itself, there holds*

$$\phi_n^{\wedge}(v; \not{p}; \gamma^*) = \delta_{k,0}, \qquad v = 2k\pi \quad (k \in \mathbb{Z}) \tag{13}$$

and, concerning the zeroes of the corresponding derivatives,

$$D^j \phi_n^{\wedge}(v; \not{p}; \gamma^*)|_{v=2k\pi} \begin{cases} = 0, & j < \min\{p_1, 2n\} \\ \neq 0, & j = \min\{p_1, 2n\} \end{cases} \quad (j \in \mathbb{N};\ k \in \mathbb{Z}). \tag{14}$$

(For all proofs, here and in what follows, see Sect. 5.) Thus conditions (3) and (4) on $\varphi^{\wedge}$ are fulfilled for $\phi_n^{\wedge}$, and that with $r = \min\{p_1, 2n\}$.

Now, with γ^* of (12) a particular combination (7) is selected as

$$\phi_n(x; \not{p}; \gamma^*) = \sum_{i=1}^{n} \gamma_i^* M_{p_i}(x), \tag{15}$$

Its most characteristic properties are

$$\phi_n(x) \equiv 0, \qquad |x| \geqslant p_n/2, \tag{16}$$

$$\phi_n(x) \in S_{p_n - 1}, \qquad |x| \leqslant p_n/2, \tag{17}$$

$$\phi_n \in C_0^{(p_1 - 2)}(\mathbb{R}). \tag{18}$$

The basic relation (16) finally confirms that the initial condition of the proposition is satisfied for (15) with finite support given by $\sigma = p_n/2$. The polynomial degree of the splines in (17) also depends upon the *maximal* index p_n of (7), whereas the order of differentiability is given by the *minimal* leading exponent p_1 of (10) (see, e.g., [4, p. 197]).

Recollecting the results there exists, at first, the following

COROLLARY. *For $f \in C^{(p_1)}(\mathbb{R})$ it holds true for any $\not p$ that*

$$\left\| \sum_{[\omega \cdot - p_n/2]+1}^{[\omega \cdot + p_n/2]} f\left(\frac{k}{\omega}\right) \phi_n(\omega \cdot - k; \not p; \gamma^*) - f(\cdot) \right\|_C \leqslant \frac{M^*}{\omega^r} \tag{19}$$

with

$$M^* = \frac{\|\phi_n\|_C \|f^{(p_1)}\|_C \, p_n^{r+1}}{p_1! \, 2^{r-2}}, \tag{20}$$

establishing a sampling sum with p_n terms and spline kernel of polynomial degree $(p_n - 1)$ given by (15)–(18).

As an immediate consequence in regard to the optimal approximation order as given in (19) and defining the "optimal linear combination" in (7) by "smallest indices" of the splines M_{p_i} (or equivalently by smallest exponents of the sinc-functions in (10), this being appropriate from the whole construction and the side conditions involved) there holds

THEOREM 2. *The optimal linear combination* (7) *containing m terms, $m \in \mathbb{N}$, which guarantees an optimal order of approximation ω^{-r} in* (5) *with corresponding m-tuples of consecutive naturals*

$$\not p^{\text{opt}} := (r, r+1, \ldots, r+m-1) \qquad (r \geqslant 2) \tag{21}$$

is given (with $\gamma^{\text{opt}} \in \mathbb{Z}^m \setminus \{0\}$ being incorporated) by

$$\begin{aligned} \phi_m(x; \not p^{\text{opt}}; \gamma^{\text{opt}}) &\equiv \phi_{m,r}^{\text{opt}}(x) \\ &:= m \binom{r+m-1}{m} \sum_{j=0}^{m-1} \frac{(-1)^j}{r+j} \binom{m-1}{j} M_{r+j}(x) \end{aligned} \tag{22}$$

with even or odd r, i.e.,

$$r = \begin{cases} 2m, & m = 1, 2, 3, \ldots, \\ 2m-1, & m = 2, 3, 4, \ldots. \end{cases}$$

The corresponding Fourier transform (10) *is characterized by*

$$\begin{aligned} \hat{\phi}_{m,r}^{\text{opt}}(v) = 1 &- \frac{1}{2^{3m} 3^m} \binom{r+m-1}{m} v^{2m} \\ &+ O(v^{2m+2}) \qquad (v \to 0). \end{aligned} \tag{23}$$

To illustrate the rather unwieldly looking general formula (22) (it definitely results in some handsome linear combinations provided r is not too large, which will be the case in the applications), the data of the first five optimal sampling kernels are given in Table I.

TABLE I

Optimal order of approximation r	Linear combination, number of terms m	Number of samples $r+m-1$	$\mathscr{P}^{\mathrm{opt}}$	γ^{opt}	Optimal kernel
2	1	2	2	1	$M_2(x)$
3	2	4	(3, 4)	(4, −3)	$4M_3(x)-3M_4(x)$
4	2	5	(4, 5)	(5, −4)	$5M_4(x)-4M_5(x)$
5	3	7	(5, 6, 7)	(21, −35, 15)	$21M_5(x)-35M_6(x)+15M_7(x)$
6	3	8	(6, 7, 8)	(28, −48, 21)	$28M_6(x)-48M_7(x)+21M_8(x)$

It should be mentioned in this connection that the optimality of $\phi_{1,2}^{\text{opt}}(x) \equiv M_2(x)$ is considered in [12, 3]. The almost optimal combination (setting $\bar{\mu} := (3, 5)$ in the corollary, thus $\bar{\gamma}^* = (\frac{5}{2}, -\frac{3}{2})$ from (12)), namely

$$\phi(x; \bar{\mu}; \bar{\gamma}^*) := \tfrac{1}{2}(5M_3(x) - 3M_5(x))$$

with 5 samples and rate ω^{-3} in (19) is discussed extensively in [3], including figures.

4. Central Factorial Numbers and $(\operatorname{sinc} x)^p$, $p \in \mathbb{N}$

The main auxiliary tool here is an utmost simple representation of the power series expansion of all natural powers of $\sin x$, thus of $\operatorname{sinc} x$, too, and that by means of central factorial numbers. For the sake of consistency the definition and some first properties to be used below are recollected from [13, pp. 213; 233] (a further, more complete list will be given in [19]).

Starting from the central factorial polynomials

$$x^{[n]} := x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right) \cdots \left(x + \frac{n}{2} - n + 1\right)$$

$$x^{[0]} := 1 \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

the central factorial numbers of 1. kind $t(n, k)$ and of 2. kind $T(n, k)$, respectively, are defined by the inverse relations

$$x^{[n]} = \sum_{k=0}^{n} t(n, k)\, x^k, \qquad x^n = \sum_{k=0}^{n} T(n, k)\, x^{[k]} \qquad (n \in \mathbb{N}_0).$$

The basic recurrences are given by

$$t(n, k) = t(n-2, k-2) - \tfrac{1}{4}(n-2)^2\, t(n-2, k),$$

$$T(n, k) = T(n-2, k-2) + \tfrac{1}{4}k^2 T(n-2, k) \qquad (n, k \geqslant 2).$$

Some particular values as well as the sign behaviour are

$$\begin{aligned} &t(n, 0) = T(n, 0) = \delta_{n,0}, && \operatorname{sgn} t(2n+1, 2k+1) \\ &t(n, k) = T(n, k) = 0 \quad (n < k), && = \operatorname{sgn} t(2n, 2k) = (-1)^{n+k}, \quad (24) \\ &t(n, n) = T(n, n) = 1, && T(n, k) > 0 \quad (0 \leqslant k \leqslant n); \end{aligned}$$

$$t(2n+1, 2k) = T(2n+1, 2k) = 0 = t(2n, 2k+1) = T(2n, 2k+1). \quad (25)$$

Two main properties, essentially applied below, are

$$k!\ T(n,k)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{k}{2}-j\right)^n \qquad (n,k\in\mathbb{N}), \tag{26}$$

$$T(2k+p,p)=\frac{1}{2^{2k}}\sum_{j=0}^{k}a_{kj}\binom{2k+p}{2k+j},$$

$$a_{k0}=\delta_{k,0},$$

$$a_{k1}=1,\quad a_{kj}=\sum_{i=j-1}^{k-1}a_{i,j-1}\binom{2k+j-1}{2k-2i}>0\ (0<j<k), \tag{27}$$

$$a_{kk}=\frac{(3k)!}{(2\cdot 3)^k\,k!}.$$

These numbers (which have received meager attention [13, p. 213] so far) enable the following

LEMMA. *The Taylor series for the powers of* sin x *and* sinc x, *respectively, are given by* $(i=\sqrt{-1})$

$$(\sin x)^p=\sum_{k=p}^{\infty}(-1)^k\,i^{p+k}\frac{2^kp!}{2^pk!}\,T(k,p)\,x^k\ \ (x\in\mathbb{R},\ p\in\mathbb{N}) \tag{28}$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}p!}{(p+2k)!}\,T(p+2k,p)\,x^{p+2k}, \tag{29}$$

$$\left(\operatorname{sinc}\frac{x}{2}\right)^p=\sum_{k=0}^{\infty}(-1)^k\frac{p!}{(p+2k)!}\,T(p+2k,p)\,x^{2k}. \tag{30}$$

One proof of (28) runs as follows ([19]; in [15] the essential last implication is missing)

$$D^k(\sin x)^p_{/x=0}=D^k\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^p_{/x=0}\qquad (k,p\in\mathbb{N})$$

$$=\frac{(-1)^k\,i^{p+k}p^k}{2^k}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{2l}{p}\right)^j$$

$$=(-1)^k\,i^{p+k}2^{k-p}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\left(\frac{p}{2}-l\right)^k$$

$$=(-1)^k\,i^{p+k}2^{k-p}p!\ T(k,p),$$

first, by the binomial theorem, and then by (26) and (24) as well. Representation (29) makes use of (25). Hence (30) is immediate.

With respect to central factorial numbers of 1. kind it is worhtwhile to note that the power series expansion for $(\arcsin x)^p$, $|x| \leqslant 1$, is identical with (28) or (29), respectively, provided (the letter) T is replaced by t (see [19]).

In connection with the importance of the central factorial numbers only one further recent property (due to S. Ries) connecting the $T(n, k)$ with the moments of the B-splines may be mentioned. It is the surprisingly simple formula

$$T(2k+n, n) = \binom{2k+n}{n} \int_{-\infty}^{\infty} x^{2k} M_n(x)\, dx \qquad (k \in \mathbb{N}_0,\ n \in \mathbb{N},\ \geqslant 2);$$

the proof follows from (9) and (30) by equating coefficients.

Last but not least, compare [11] for another realization of a well-behaved sampling theorem involving B-splines as well as—in a decisive way, too—central factorial numbers (of 2. kind); in this respect see also [10].

5. Proofs

Concerning Theorem 1, the general condition (11) as applied to (10), in short,

$$\hat{\phi_n}(v; \not{p}; \gamma^*) := 1 + k_{2n} v^{2n} + O(v^{2n+2}) \qquad (v \to 0), \tag{31}$$

leads immediately, by using (30), to the system of linear equations

$$\begin{aligned} &\text{(i)} \quad \sum_{i=1}^{n} \gamma_i^* = 1, \\ &\text{(ii)} \quad \sum_{i=1}^{n} \frac{(p_i)!}{(2k+p_i)!} T(2k+p_i, p_i)\, \gamma_i^* = 0 \qquad (k = 1, 2, \ldots, n-1). \end{aligned} \tag{32}$$

The set of conditions (32), (ii) will be simplified as follows. Introducing Stirling numbers of 1. kind $s(j, k)$ (see, e.g., [13, p. 90]),

$$\begin{aligned} x(x-1)\cdots(x-j+1) &\equiv \prod_{k=0}^{j-1} (x-k) \\ &= \sum_{k=1}^{j} s(j, k)\, x^k, \qquad s(j, j) = 1 \quad (j \in \mathbb{N}), \end{aligned}$$

in combination with (27) the coefficients of (32), (ii) are rewritten (with general $p \in \mathbb{N}$, at first) as

$$\frac{p!}{(2k+p)!} T(2k+p, p) = \frac{1}{2^{2k}} \sum_{j=1}^{k} a_{kj} \frac{1}{(2k+j)!} \sum_{r=1}^{j} s(j, r)\, p^r.$$

So (32), (ii) is equivalent to

$$\sum_{j=1}^{k} a_{kj} \frac{1}{(2k+j)!} \sum_{r=1}^{j} s(j,r) \sum_{i=1}^{n} p_i^r \gamma_i^* = 0 \qquad (1 \leqslant k \leqslant n-1). \tag{33}$$

For $k=1$ it is easily verified that (33) is satisfied if and only if $\sum_{i=1}^{n} p_i \gamma_i^* = 0$. By induction it then follows that (32),(ii) holds if and only if

$$\sum_{i=1}^{n} p_i^k \gamma_i^* = 0 \qquad (1 \leqslant k \leqslant n-1). \tag{34}$$

Thus the inhomogeneous (only a single 1 on the right-hand side) system (32),(i), (34) for the γ_i^* $(1 \leqslant i \leqslant n)$, in short

$$\sum_{i=1}^{n} p_i^k \gamma_i^* = \delta_{k,0} \qquad (0 \leqslant k \leqslant n-1)$$

has to be solved. However, since all p_i are different by assumption, the system determinant

$$P := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & & p_n \\ \vdots & & & \vdots \\ p_1^{n-1} & p_2^{n-1} & \cdots & p_n^{n-1} \end{vmatrix} = \prod_{1 \leqslant \nu < \mu \leqslant n} (p_\mu - p_\nu) \neq 0 \tag{35}$$

is a Vandermonde determinant with well-known value as given. Solving for γ_i^* it follows (by the usual techniques) that

$$\gamma_i^* = \frac{1}{P} \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ p_1 & & p_{i-1} & 0 & p_{i+1} & & p_n \\ \vdots & & & \vdots & & & \vdots \\ p_1^{n-1} & \cdots & p_{i-1}^{n-1} & 0 & p_{i+1}^{n-1} & \cdots & p_n^{n-1} \end{vmatrix} \qquad (1 \leqslant i \leqslant n)$$

$$= \frac{(-1)^{i+1}}{P} \begin{vmatrix} p_1 & \cdots & p_{i-1} & p_{i+1} & \cdots & p_n \\ \vdots & & \vdots & & & \vdots \\ p_1^{n-1} & \cdots & p_{i-1}^{n-1} & p_{i+1}^{n-1} & \cdots & p_n^{n-1} \end{vmatrix}$$

$$= \frac{(-1)^{i+1}}{P} \prod_{\substack{\mu=1 \\ \mu \neq i}}^{n} p_\mu \cdot \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ p_1 & & p_{i-1} & p_{i+1} & & p_n \\ \vdots & & \vdots & & & \vdots \\ p_1^{n-2} & \cdots & p_{i-1}^{n-2} & p_{i+1}^{n-2} & \cdots & p_n^{n-2} \end{vmatrix}$$

$$= \frac{(-1)^{i+1}}{P} \prod_{\substack{\mu=1 \\ \mu \neq i}}^{n} p_\mu \cdot \prod_{\substack{1 \leqslant \nu < \mu \leqslant n \\ \mu,\nu \neq i}} (p_\mu - p_\nu) \tag{36}$$

since the latter determinant is again of type (35).

Finally, by simple calculations, (36) reduces to (12), showing simultaneously that all the γ_i are rationals.

The coefficient k_{2n} of (31) again follows from (30) using (27):

$$\begin{aligned} k_{2n} &= (-1)^n \sum_{i=1}^{n} \frac{(p_i)!}{(2n+p_i)!} T(2n+p_i, p_i)\gamma_i^* \\ &= (-1)^n \frac{1}{2^{2n}} \sum_{j=1}^{n} a_{nj} \frac{1}{(2n+j)!} \sum_{r=1}^{j} s(j,r) \sum_{i=1}^{n} p_i^r \gamma_i^* \\ &= (-1)^n \frac{1}{2^{2n}} a_{nn} \frac{1}{(3n)!} s(n,n) \sum_{i=1}^{n} p_i^n \gamma_i^*, \end{aligned} \tag{37}$$

since, in view of (34), only the term for $j=n$ does not vanish. Now noting (36),

$$\begin{aligned} \sum_{i=1}^{n} \gamma_i^* p_i^n &= \sum_{i=1}^{n} \frac{(-1)^{i+1}}{P} \prod_{\substack{\mu=1 \\ \mu\neq i}}^{n} p_\mu \cdot \prod_{\substack{1\leqslant \nu<\mu\leqslant n \\ \nu,\mu\neq i}} (p_\mu - p_\nu)\cdot p_i^n \\ &= \frac{1}{P} \prod_{\mu=1}^{n} p_\mu \sum_{i=1}^{n} (-1)^{i+1} p_i^{n-1} \prod_{\substack{1\leqslant \nu<\mu\leqslant n \\ \nu,\mu\neq i}} (p_\mu - p_\nu) \\ &= (-1)^{n+1} \prod_{\mu=1}^{n} p_\mu \left\{ \frac{1}{P} \sum_{i=1}^{n} (-1)^{n+i} p_i^{n-1} \prod_{\substack{1\leqslant \nu<\mu\leqslant n \\ \nu,\mu\neq i}} (p_\mu - p_\nu) \right\} \\ &= (-1)^{n+1} \prod_{\mu=1}^{n} p_\mu, \end{aligned} \tag{38}$$

since the expression in brackets equals 1 by expanding the determinant of (35) according to the last row. Combining (37) and (38) together with a_{nn} of (27) yields

$$k_{2n} = -\frac{1}{2^{3n}3^n n!} \prod_{\mu=1}^{n} p_\mu; \tag{39}$$

thus the constant (always negative!) of (11) has as main factor the product of all exponents p_i of (10).

The fact that conditions (13) and (14) are all satisfied is readily read off from (11) and the construction per se; it only has to be observed that the essential restriction $j < \min\{p_1, 2n\}$ depends upon the smallest exponent p_1 of the combination (10) as well as upon the exponent of the first nonconstant term in the expansion (11).

As for the proof of Theorem 2, some comments: Concerning the optimal case (for m terms), i.e., order $r = 2m$ ($m \in \mathbb{N}$) and thus the particular vector of m consecutive exponents (21), i.e.,

$$\not{p} = (2m, 2m+1, \ldots, 3m-1), \qquad p_i := 2m - 1 + i \quad (1 \leqslant i \leqslant m)$$

it is concluded for the vector γ^* of the coefficients (12) that

$$\gamma_i^* = (-1)^{m+1} \frac{\frac{1}{p_i} \prod_{\mu=1}^{m} p_\mu}{\prod_{\mu=1}^{m}{}' (p_i - p_\mu)} \qquad (1 \leqslant i \leqslant m) \tag{40}$$

$$= (-1)^{m+1} \left(\frac{2}{3} \frac{m!}{(2m-1+i)} \binom{3m}{m} \right) \Bigg/ (-1)^{m-i} \frac{1}{i} \frac{m!}{\binom{m}{i}}, \tag{41}$$

indicating the separate (straightforward) evaluation of the numerator and the denominator of (40). Finally, (41) yields

$$\gamma_i^* = (-1)^{i+1} \frac{i}{2m-1+i} \binom{m}{i} \binom{3m-1}{m} \qquad (1 \leqslant i \leqslant m). \tag{42}$$

Moreover, by an argument of elementary number theory it is seen that all γ_i^* are integers ($\neq 0$), alternating in sign. Substituting (42) into (7) delivers (22) in form of

$$\phi_{m,2m}^{\text{opt}}(x) = m \binom{3m-1}{m} \sum_{j=0}^{m-1} \frac{(-1)^j}{2m+j} \binom{m-1}{j} M_{2m+j}(x).$$

Second, the same procedure as before in case (also with m terms) of odd order $r = 2m-1$ ($m \geqslant 2$) yields

$$\phi_{m,2m-1}^{\text{opt}}(x) = m \binom{3m-2}{m} \sum_{j=0}^{m-1} \frac{(-1)^j}{2m-1+j} \binom{m-1}{j} M_{2m-1+j}(x),$$

so that the unified version (22) holds in any of the two cases. The constant in the corresponding Taylor expansion (23) is built up from (39) using (41).

6. Concluding Remarks

It should be mentioned that the above (theoretical) results have been checked (on the CYBER 175 of the Rechenzentrum, RWTH) for various cases (e.g., of Table I) using reasonable (small) values of ω in (19) or in Theorem 2. These tests emphasize that the results are of real practical importance; at the same time they show that the (bad) constant (20) is, in fact, (merely) a theoretical one. (A discussion of these numerical experiments will be published elsewhere.)

Parts of these investigations have been announced on the occasion of the 5th Aachen Colloquium on Mathematical Methods in Signal Processing, September 1984 (see [6] for $n=2$).

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